Earth-like circulation
and
Venus-like circulation

*Hiroki Kashimura (ISAS/JAXA)

(*Formerly known as Hiroki Yamamoto)
Earth-like circulation and Venus-like circulation in an Idealized Axisymmetric model

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Introduction
Observation | Venus

- Venus’ size and mass are similar to Earth’s.
  - radius ~ 6052 km, gravity ~ 8.9 m/s²

- Venus rotates very slow.
  - rotation period ~ 243 (Earth) days

- Zonal wind in the Venus atmosphere reaches ~ 100 m/s.
  - This is **60 times faster** than the planetary rotation at the equator!

Superrotation

Schubert (1983)
Mystery of superrotation

• Eddy viscosity and surface friction slow down the atmosphere.

• Angular momentum must be pumped up to the atmosphere by some mechanism.

Hypotheses

‣ dayside-nightside circulation (e.g., Schubert & Whitehead, 1969)

‣ gravity waves (e.g., Fels & Linzen, 1974)

‣ meridional circulation (e.g., Gierasch, 1975)
Gierasch (1975) mechanism

- Gierasch (1975) assumed
  - symmetries about the rotation axis and the equator
  - Hadley cell expanding from the equator to the pole
  - infinitely large horizontal diffusion

⇒ always solid body rotation = constant angular vel.

Sketch of Gierasch mechanism
• Gierasch stated that, for the mechanism to work,
  - relaxation time of the horizontal diffusion must be
    ▶ much shorter than that of vertical diffusion
    ▶ and turnover time of meridional circulation.

• Matsuda (1980, 1982) explored
  - parameter dependency of \( U \),
  - dominant moment balance,
  - multiple equilibrium solutions,
for both infinite and finite hor. diff. cases,
using a highly truncated low-order spectral model.
In my study

We theoretically and numerically explore the strength of the superrotation maintained by the Gierasch mechanism using an idealized system.

In Matsuda (1980, 1982)

• advection of potential temperature was ignored,

• math was complicated because of mode equations,

• theoretical results were not verified by numerical experiments with high orders.

In this study, we

• treat the meridional temp. difference as an internal variable,

• develop a theoretical model expressed by algebraic equations for superrotation strength

• verify the theoretical solution by numerical experiments
Basic equations

- primitive equations
- dry Boussinesq fluid
- Newtonian heating and cooling
- axisymmetric with strong horizontal diffusion of momentum

$$\begin{align*}
\frac{\partial u}{\partial t} + \frac{v}{a} \frac{\partial u}{\partial \phi} + w \frac{\partial u}{\partial z} - \frac{uv \tan \phi}{a} - 2\Omega v \sin \phi &= \nu_H D_H(u) + \nu_V \frac{\partial^2 u}{\partial z^2} \\
\frac{\partial v}{\partial t} + \frac{v}{a} \frac{\partial v}{\partial \phi} + w \frac{\partial v}{\partial z} + \frac{u^2 \tan \phi}{a} + 2\Omega u \sin \phi &= -\frac{1}{a} \frac{\partial \Phi}{\partial \phi} + \nu_H D_H(v) + \nu_V \frac{\partial^2 v}{\partial z^2} \\
\frac{\partial \Phi}{\partial z} &= g \frac{\theta - \Theta_0}{\Theta_0} \\
\frac{\partial \theta}{\partial t} + \frac{v}{a} \frac{\partial \theta}{\partial \phi} + w \frac{\partial \theta}{\partial z} &= -\frac{\theta - \theta_e}{\tau} + \kappa_V \frac{\partial^2 \theta}{\partial z^2} \quad \theta_e \equiv \Theta_0 \left[1 - \Delta_H \left(\sin^2 \phi - \frac{1}{3}\right)\right] \\
\frac{1}{a \cos \phi} \frac{\partial}{\partial \phi} (v \cos \phi) + \frac{\partial w}{\partial z} &= 0
\end{align*}$$
Boundary conditions

• Top: free-slip, no mass or heat flux,

\[ \frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = w = \frac{\partial \theta}{\partial z} = 0 \quad (\text{top: } z = H) \]

• Bottom: no-slip, no mass or heat flux

\[ u = v = w = \frac{\partial \theta}{\partial z} = 0 \quad (\text{bottom: } z = 0) \]

• Equator and pole: no mass, momentum, or heat flux

\[ v = \frac{\partial u}{\partial \phi} = \frac{\partial w}{\partial \phi} = \frac{\partial \theta}{\partial \phi} = 0 \quad \left( \text{eq. : } \phi = 0, \text{pole: } \phi = \frac{\pi}{2} \right) \]
How to develop a theoretical model

Basic equations are nonlinear PDEs, which we cannot solve analytically.

We assume
- steady state: \( \partial / \partial t = 0 \)
- spacial structure of some variables

spacial integral, relation of scales

**Theoretical model** which is a set of algebraic equations for **four nondimensional variables**
How to develop a theoretical model

**Theoretical model** which is a set of algebraic equations for **four nondimensional variables**

- **Superrotation strength**
  - \[ S \equiv \frac{U}{a\Omega} \]
  - \[ R_{vB} \equiv \frac{V_B}{a\Omega} \]
  - \[ R_{vT} \equiv \frac{V_T}{a\Omega} \]
  - \[ \beta \equiv \frac{\Delta \Theta}{\Theta_0 \Delta H} \]

  **Meridional mean zonal wind at the top**

- low-order system
- enable to solve analytically
- easy to see the parameter dependency
Assumption: spacial structure of variables

- zonal wind at the top:

- meridional wind at the top:

- meridional wind above the surface:

- vertical mean of potential temp.:

- meridional integral of angular momentum:
3.4 Development of the theoretical model

**Zonal momentum eq.**

Above the surface: \( \frac{8}{15} a \Omega V_B \approx \nu_V \frac{U \alpha \pi^2}{2H^2} \)

At the top: \(- \frac{8}{15} \left( \frac{3}{2} U + a \Omega \right) V_T \approx -\nu_V \frac{U \alpha \pi^2}{2H^2} \)

Non-dimensionalize: \( R_{vB} \approx \pi^2 E_V S \)

Non-dimensionalize: \( R_{vT} \approx \pi^2 E_V \left( \frac{S}{1 + S} \right) \)

**Thermodynamic eq.**

Vertical mean: \( \frac{1}{aH \cos \phi} \int_0^H \frac{\partial}{\partial \phi} (v \theta \cos \phi) dz \approx -\frac{(\Theta_0 \Delta_H - \Delta \Theta)}{\tau} \left( \sin^2 \phi - \frac{1}{3} \right) \)

Relation of scales: \( \frac{R_{vT} + R_{vB}}{2} \approx \frac{1}{\tau \Omega} \left( \frac{1}{\beta} - 1 \right) \)

Non-dimensionalize: \( S^2 + 2S \sim 2\beta R_T - 20\mathcal{E}_H (R_{vT} + R_{vB}) \)

**Ev: vertical Ekman num., \( E_H: \) horizontal Ekman num., \( R_T: \) external thermal Rossby num.**

\[ \equiv \nu_V / (H^2 \Omega) \quad \equiv \nu_H / (a^2 \Omega) \quad \equiv gH \Delta_H / (a^2 \Omega^2) \]
algebraic equations for $S$, $R_{vB}$, $R_{vT}$, $\beta$

$$R_{vB} = \pi^2 E_V S$$

$$R_{vT} = \pi^2 E_V \left( \frac{S}{1 + S} \right)$$

$$\frac{R_{vT} + R_{vB}}{2} = \frac{1}{\pi \Omega} \left( \frac{1}{\beta} - 1 \right)$$

$$S^2 + 2S = 2\beta R_T - 20E_H(R_{vT} + R_{vB})$$

Quintic equation for $S$

$$\left[ S^2 + 2S + BS \left( \frac{2 + S}{1 + S} \right) \right] \left[ \frac{AS}{2} \left( \frac{2 + S}{1 + S} \right) + 1 \right] = 2R_T$$

$$A \equiv \pi^2 \tau \Omega E_V \quad B \equiv 20\pi^2 E_H E_V$$
Quintic equation for $S$

\[
\left[ S^2 + 2S + BS \left( \frac{2 + S}{1 + S} \right) \right] \left[ \frac{AS}{2} \left( \frac{2 + S}{1 + S} \right) + 1 \right] = 2R_T
\]

This eq. has only one positive solution. The positive solution estimates the superrotation strength.

It depends only on three external parameters

\[ A \equiv \pi^2 \tau \Omega E_V = \pi^2 \frac{\tau}{H^2/\nu_V} \quad : \text{the ratio of the radiative relaxation time to the timescale for vertical eddy diffusion of momentum} \]

\[ B \equiv 20\pi^2 E_H E_V = 5 \left( \frac{2\pi/\Omega}{\sqrt{a^2/\nu_H}(H^2/\nu_V)} \right)^2 \quad : \text{the ratio of the rotation period to the geometric mean of the timescales for horizontal and vertical eddy diffusion} \]

\[ R_T \equiv \frac{gH \Delta_H}{a^2 \Omega^2} \quad : \text{the external thermal Rossby number} \]
Approximation of the quintic eq.  

\[
\begin{bmatrix}
S^2 + 2S + BS \left(\frac{2 + S}{1 + S}\right)
\end{bmatrix} \approx 2 \cdot 10^2
\]

\[
1 < \left(\frac{2}{1 - S}\right)
\]

\[
(S^2 + 2S + BCS) \left(\frac{ACS}{2} + 1\right) \approx 2R
\]

Cubic equation for \( S \) Solvab

\[
\frac{AC}{2} S^3 + \left(1 + AC + \frac{ABC^2}{2}\right) S^2 + (2 + BC) S \approx 2R_T
\]
Further approximation

From the cubic eq., $S$ can be approximated as...

$$\frac{AC}{2}S^3 + \left(1 + AC + \frac{ABC^2}{2}\right)S^2 + (2 + BC)S \approx 2RT$$

### Table

<table>
<thead>
<tr>
<th>Case</th>
<th>Condition</th>
<th>Approximation</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td>$B \ll 1$</td>
<td>$A \ll 1$</td>
</tr>
<tr>
<td>(b)</td>
<td></td>
<td>$A \gg 1$</td>
</tr>
<tr>
<td>(c)</td>
<td>$AB \ll 1$</td>
<td>$2RT$</td>
</tr>
<tr>
<td>(d)</td>
<td>$B \gg 1$</td>
<td>$\frac{2R_T}{BC}$</td>
</tr>
</tbody>
</table>

### Diagram

The diagram illustrates the approximation of $S$ for different conditions of $A$, $B$, and $C$. Each condition corresponds to a specific approximation formula, as shown in the table above.
<table>
<thead>
<tr>
<th></th>
<th>(a)</th>
<th>(b)</th>
<th>(c)</th>
<th>(d)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$B \ll 1$</td>
<td>$A \gg 1$</td>
<td>$AB \ll 1$</td>
<td>$AB \ll 1$</td>
</tr>
<tr>
<td></td>
<td>$A \ll 1$</td>
<td>$A \gg 1$</td>
<td>$2R_T \left( \frac{2R_T}{AC} \right)^{\frac{1}{3}}$</td>
<td>$2R_T \left( \frac{2R_T}{ABC^2} \right)^{\frac{1}{3}}$</td>
</tr>
<tr>
<td></td>
<td>$\sqrt{2R_T}C_1$</td>
<td>$\sqrt{2R_T}G_0$</td>
<td>$\sqrt{2R_T}H_1$</td>
<td>$\sqrt{2R_T}H_0$</td>
</tr>
<tr>
<td></td>
<td>$\frac{2}{A^2C^2}$</td>
<td>$2AC$</td>
<td>$\frac{B^2C^2}{2}$</td>
<td>$\frac{2}{A^2C^2}$</td>
</tr>
<tr>
<td></td>
<td>$\frac{2}{AC}$</td>
<td>$\frac{2}{AC}$</td>
<td>$\frac{B^2C^2}{2}$</td>
<td>$\frac{2}{AC}$</td>
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<tr>
<td></td>
<td>$\frac{B}{A}$</td>
<td>$\frac{B}{A}$</td>
<td>$\frac{B}{A}$</td>
<td>$\frac{AB^3C^4}{4}$</td>
</tr>
</tbody>
</table>

returning to the meridional eq. style

$$S^2 + 2S + BCS \approx 2R_T(\beta) = \left( \frac{ACS}{2} + 1 \right)^{-1} \frac{\Delta \Theta}{\Theta_0 \Delta H}$$

which is dominant?

1. cyclostrophic balance [C]
2. geostrophic balance [G]
3. horizontal diff. balance [H]
Numerical experiments
• Time-integration was performed from motionless state.
• We obtain steady or statistically steady numerical solutions.
• External parameters are...

<table>
<thead>
<tr>
<th></th>
<th>(A)</th>
<th>(B)</th>
<th>(E_V)</th>
<th>(\tau\Omega)</th>
<th>(E_H)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td>(\pi^2 \times 10^{-2})</td>
<td>(2\pi^2 \times 10^{-2})</td>
<td>(10^{-3})</td>
<td>10</td>
<td>1</td>
</tr>
<tr>
<td>(b)</td>
<td>(\pi^2)</td>
<td>(2\pi^2 \times 10^{-2})</td>
<td>(10^{-3})</td>
<td>(10^3)</td>
<td>1</td>
</tr>
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<td>1</td>
<td>(10^2)</td>
</tr>
<tr>
<td>(d)</td>
<td>(\pi^2 \times 10^{-1})</td>
<td>(2\pi^2)</td>
<td>(10^{-3})</td>
<td>(10^2)</td>
<td>(10^2)</td>
</tr>
</tbody>
</table>

For each combination,

\[ R_T = 10^n \ (n = -2, -1, 0, \ldots, 5) \] is calculated.

\[ A = \pi^2 \tau \Omega E_V, \ B = 20\pi^2 E_H E_V \]
Numerical superrotation strength

Meridional mean zonal wind at the top

\[ S_n = \frac{U}{a\Omega} \]

numerical superrotation strength

Nondimensional zonal wind field

\[ \left( \frac{u}{a\Omega} \right) \]
Theoretical vs. Numerical

(a) $S_n$ : numerical
- Solid curve: solution of the quintic equation

$$S^2 + 2S + BS \left( \frac{2 + S}{1 + S} \right) \left\{ \frac{AS}{2} \left( \frac{2 + S}{1 + S} \right) + 1 \right\} = 2R_T.$$ 

- $S_n$ : numerical
- Steady state: •
- Statistically-steady state: ○

Solid curves denote the theoretical solution $S^t$, the positive solution of the quintic equation. The locations of $R_T$ and $R_T'$ are indicated by the vertical lines. The types of the dominant dynamical balance are indicated by $C_1$, $C_0$, $G_1$, $G_0$, $H_1$, and $H_0$. 

Now, let us regard (21), (23), (29), and (34) as a set of algebraic equations, that is, simultaneous equations whose unknowns are $S$, $R_vB$, $R_vT$, and $\beta$. Eliminating $R_vB$, $R_vT$, and $\beta$ from the equations, we obtain

$$S^2 + 2S + BS \left( \frac{2 + S}{1 + S} \right) \left\{ \frac{AS}{2} \left( \frac{2 + S}{1 + S} \right) + 1 \right\} = 2R_T,$$

where $A \equiv \epsilon E^* V$ and $B \equiv 20 E_H E^* V$ are positive constants.

This quintic equation has only one positive solution (see Appendix for the demonstration), which is physically valid. The other solutions are complex or negative. Complex solutions have no physical meaning in this context. Negative solutions mean that the zonal wind rotates retrograde. According to (21) and (23), in that case, $R_vB$ and $R_vT$ must be negative; this means that meridional circulation must be indirect cell. Thus, $\beta$ must be $\beta > 1$ or $\beta < 0$ according to (29). When $\beta > 1$, the indirect circulation makes the atmosphere statically unstable in the mid-latitudes; because a hot air will be advected poleward near the surface and a cold air will be advected equatorward near the top. When $\beta < 0$, the atmosphere in the low-latitudes will be keep heating by both Newtonian heating and thermal advection, and the atmosphere in the high-latitudes will be keep cooling. In both cases, the atmosphere will not achieve a steady state, which we assumed at the beginning of this section. Therefore, negative solutions are not physically valid.

In this subsection, we simplify the quintic equation (35) to a cubic equation, which can be solved analytically. Then, we present a simpler expression of the solution depending on the parameters and the parametric dependence of the dominant dynamical balance in the steady state vs. statistically steady state.
Solution of the quintic equation estimates superrotation strength of the numerical solutions!
To Earth-like circulation
5. Basic equations

- primitive equations
- dry Boussinesq fluid
- Newtonian heating and cooling
- axisymmetric with strong horizontal diffusion without horizontal diffusion

\[
\begin{align*}
\frac{\partial u}{\partial t} + \frac{v}{a} \frac{\partial u}{\partial \phi} + w \frac{\partial u}{\partial z} - \frac{uv \tan \phi}{a} - 2\Omega v \sin \phi &= \nu_H \frac{\partial^2 u}{\partial z^2} \\
\frac{\partial v}{\partial t} + \frac{v}{a} \frac{\partial v}{\partial \phi} + w \frac{\partial v}{\partial z} + \frac{u^2 \tan \phi}{a} + 2\Omega u \sin \phi &= -\frac{1}{a} \frac{\partial \Phi}{\partial \phi} + \nu_H \frac{\partial^2 v}{\partial z^2} \\
\frac{\partial \Phi}{\partial z} &= g \frac{\theta - \Theta_0}{\Theta_0} \\
\frac{\partial \theta}{\partial t} + \frac{v}{a} \frac{\partial \theta}{\partial \phi} + w \frac{\partial \theta}{\partial z} &= -\frac{\theta - \theta_e}{\tau} + \kappa_V \frac{\partial^2 \theta}{\partial z^2} \quad \theta_e \equiv \Theta_0 \left[ 1 - \Delta_H \left( \sin^2 \phi - \frac{1}{3} \right) \right] \\
\frac{1}{a \cos \phi} \frac{\partial}{\partial \phi} (v \cos \phi) + \frac{\partial w}{\partial z} &= 0
\end{align*}
\]
5.2 Held and Hou (1980) model

• Axisymmetric theoretical model of the Hadley circulation of the Earth.

**Theory**

**Width of Hadley circulation**

\[ \phi_H = \left(\frac{5}{3} R_T \right)^{\frac{1}{2}} \]

Hou (1984)

\[ \sin \phi_H = \begin{cases} \left(\frac{5}{3} R_T \right)^{\frac{1}{2}} & (R_T \ll 1) \\ 1 - \frac{3}{8 R_T} & (R_T \gg 1) \end{cases} \]

**Zonal wind at the top**

| \( u_M = a \Omega \frac{\sin^2 \phi}{\cos \phi} \) | \( (0 \leq \phi < \phi_H) \)
| \( u_E = a \Omega \left[ \left(1 + \frac{2R_T z}{H} \right)^{\frac{1}{2}} - 1 \right] \cos \phi \) | \( (\phi_H \leq \phi \leq \pi/2) \)

**Thermal wind with respect to \( \theta_e \)**

**Numerical solution**

*streamfunction*  

*zonal wind*

---

Held and Hou (1980)
How solution changes?

From Gierasch mechanism solution

\[ E_H : \text{large} \]

\[ E_H = 0 \]

to Held-Hou solution

Numerical experiments
Equatorial jet (solid-body rotation)

mid-latitude jet

zonal wind at the top

zonal wind
Zonal wind at the top

- Equatorial jet sol.
  - close to solid-body

- Mid-latitude jet sol.
  - $u \approx u_{HH}$

- Held-Hou model sol.
  - $u \approx u_{HH}$

$R_T = 10^{-1}$

$B = 2\pi^2 \times \begin{cases} 10^{-3} \\ 10^{-4} \\ 10^{-5} \\ 10^{-6} \\ 10^{-7} \\ 10^{-8} \end{cases}$

Large

$\downarrow$

Hor.

Dif.

Small

$u_{HH}$
We explored the strength of the superrotation maintained by the Gierasch mechanism in an idealized axisymmetric boussinesq fluid model with strong horizontal diffusion.

PDEs assuming spatial structure of variables

Quintic eq. for $S$

numerical solution

positive solution

Cubic eq. for $S$

numerical solution

solid-body rotation

Venus-like?

Held-Hou model solution

Earth-like?
