Magnetic Rossby waves in the Earth’s core

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Waves in the Earth’s fluid core

Waves provide us with information about the ‘invisible’ system

• **torsional Alfven waves** (e.g. Braginsky 1967, Zatman & Bloxham 1997)
  – axisymmetric, travelling in radius \( r \)
  – \( \sim 6 \) yrs traveltime: \( B_s > \sim 2 \) mT (Gillet et al. 2010, 2015)

• **axisymmetric MAC oscillations** (e.g. Braginsky 1993)
  – in a thin, stably stratified layer at the top of the core?
  – \( \sim 60 \) yrs geomagnetic variation: \( H \sim 140 \) km? (Buffett 2014)

• **slow magnetic Rossby waves** (e.g. Hide 1966, Acheson 1978)
  – nonaxisymmetric, travelling in azimuth \( \phi \)
  – \( \sim 300 \) yrs westward drift: \( B\phi \sim 10 \) mT? (Hori et al. 2015)

• **(fast magnetic) Rossby waves in a thin stable layer** (e.g. Braginsky 1984)
  – \( \sim 6 \) yrs westward drift? (Chulliat et al. 2015)
  – in the solar tachocline also?: \( \sim 2 \) yrs westward? (McIntosh et al. 2017)
An axisymmetric mode: torsional Alfven waves

- A special class of Alfven waves (Braginsky 1970; also Roberts & Aurnou 2012)
  - the azimuthal momentum eq on cylindrical surfaces in the magnetostrophic balance gives a steady state (Taylor 1963)
  - cylindrical perturbations on the state

\[
\frac{\partial^2 \langle u'_\phi \rangle}{\partial t^2} = \frac{1}{s^3 h(\bar{\rho})} \frac{\partial}{\partial s} \left( s^3 h(\bar{\rho}) U_A^2 \frac{\partial}{\partial s} \langle u'_\phi \rangle \right)
\]

» travel in radius s with the the z-mean Alfven speed \( U_A = \langle B_s^2 \rangle / \langle \rho \rangle \mu_0 \)^{1/2}
An axisymmetric mode: torsional Alfvén waves

- A special class of Alfvén waves (Braginsky 1970; also Roberts & Aurnou 2012):
  - the azimuthal momentum eq on cylindrical surfaces in the magnetostrophic balance gives a steady state (Taylor 1963)
  - cylindrical perturbations on the state
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    \frac{\partial^2 \langle u_\phi' \rangle}{\partial t^2} = \frac{1}{s^3 h(\bar{\rho})} \frac{\partial}{\partial s} \left( s^3 h(\bar{\rho}) U_A^2 \frac{\partial \langle u_\phi' \rangle}{\partial s} \right)
    \]
    \[
    \rightarrow \text{travel in radius s with the the z-mean Alfvén speed } U_A = \langle B_s^2 \rangle / \langle \rho \mu_0 \rangle^{1/2}
    \]

- Data:
  - probably responsible for 6-7 year variations
    \[
    \rightarrow \text{can account for the 6 year LOD change}
    \]
  - the observed wave speed is used to infer the field strength within the core
    \[
    \rightarrow \langle B_s^2 \rangle^{1/2} \geq 2 \text{ mT}
    \]
    \[
    \rightarrow \text{better fits with the scaling law}
    \]

(Gillet et al. 2010)
Nonaxisymmetric waves in the core?

- Possibly related to the geomagnetic westward drift
  - the nonaxisymmetric part of the field moving in azimuth
    - significant in the Atlantic hemisphere: period ~ $3 \times 10^2$ yrs
  - probably a mixture of flow advection (Bullard+ 1950) and wave propagation (Hide 1966)
  - How can we separate the signal due to waves?

Nonaxisymmetric part of Br at the surface of the core at the equator / 40° S (gufm1: Finlay & Jackson 2003)
Magnetic Rossby waves

• **Key ingredients** (Hide 1966; Acheson 1978; also Hori et al. 2015):
  
  – axial vorticity equation in a quasi-magnetostrophic balance (\(\Lambda=O(1)\); Ro, E<<1)
    
    \[
    \rho \frac{\partial \zeta'}{\partial t} - 2\rho \Omega \frac{\partial u_z'}{\partial z} = \hat{e}_z \cdot \nabla \times (j' \times \vec{B})
    \]
  
    coupled with the induction equation
    
    \[
    \frac{\partial b'}{\partial t} = \vec{B} \cdot \nabla u'
    \]
  
  – spherical geometry (topographic \(\beta\)–effect)
  
  – almost independent of \(z\) (quasi-geostrophic)
  
  – azimuthal length scales shorter than radial ones

• **Dispersion relations about a mean flow:**
  
  with a form of \(e^{i(m\phi - \omega t)}\)
  
  \[
  \hat{\omega} = \hat{\omega}_R \left[ \frac{1}{2} \pm \frac{1}{2} \sqrt{1 + \frac{\hat{\omega}_M^2}{\hat{\omega}_R^2}} \right]
  \]
  
  where Rossby and Alfven frequencies
  
  \[
  \hat{\omega}_R = \frac{2\Omega s^2}{(r_o^2 - s^2)m} \quad \hat{\omega}_M^2 = \frac{m^2 \langle B^2 \rangle}{\rho \mu_0 s^2}
  \]

  a QG eigenfunction for \(B_\phi = B_0 \sin \phi\)
  
  in a meridional section
  
  (after Malkus 1967)
Magnetic Rossby waves (cont’d)

- **Fast modes:**
  - $\omega \to + \omega_R (1 + \omega_M^2 / \omega_R^2)$ in the limit $\omega_M^2 / \omega_R^2 \ll 1$
  - essentially (nonmag) Rossby waves (Busse 1986)
  - travelling progradely (eastward) with timescales of $O$(months) in the fluid core

- **Slow modes:**
  - $\omega \to - \omega_M^2 / \omega_R$ in the limit $\omega_M^2 / \omega_R^2 \ll 1$
    \[ \hat{\omega}_{MR} = - \frac{\hat{\omega}_M^2}{\hat{\omega}_R} = - \frac{m^3 r_0^2 - s^2 B_\phi^2}{2 \rho \mu_0 \Omega_s^4} \]
  - travelling retrogradely (westward) along the toroidal field $B_\phi$ on timescales of $O(10^2 \text{ years})$
    - cf. torsional Alfven waves along $B_s$
  - highly dispersive
  - the governing equations (Cartesian)
    \[ \frac{\partial j_z'}{\partial t} = \frac{B_{0z}}{\mu_0} \frac{\partial \xi_z'}{\partial x} \]
    \[ - \frac{4 \rho \Omega \chi}{L} u_y' = \frac{B_{0x}}{\mu_0} \frac{\partial j_z'}{\partial x} \]

(Hori, Takehiro & Shimizu, 2014)
Waves hint at strong-field dynamos?

- **Linear, rotating magnetoconvection**
  (e.g. Chandrasekahr 1961, Fearn 1979; also Zhang & Schubert 2000):
  - as magnetic field is strengthened to $\Lambda = O(1)$, the thermal stability $Ra_{crit}$, the preferred wavenumber $k_{crit}$, and wave frequency $\omega_{crit}$ drop
  - dynamos hypothesized in the regime: ‘strong-field’ dynamos (e.g. Roberts 1978)
  - Note: all three effects not necessarily depend on the background magnetic field, boundary conditions, etc.

![Diagram showing the relationship between magnetic field strength and Rayleigh number](image)

- Field strength $B^2$ or $\Lambda$
- Flow vigor $U$ or $Rm$
- Rayleigh number $Ra$
- $Ra_{d}$ and $Ra_{d}^{mag}$
- $Ra_{c}$ and $Ra_{c}^{mag}$
- $Rm_{d}$ and $Rm_{d}^{mag}$
- Strong field dynamo
- Weak field dynamo

$Ra_{d}^{mag} = O(E^{-4/3})$

$Ra_{c}^{mag} = O(E^{-1})$

$\Lambda = O(1)$

$\Lambda = O(E)$
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- **Convection-driven spherical dynamos** likely approaching the regime (e.g. Yadav et al. 2016; Dormy 2016)
  - force balances
  - **flow properties?** (vigor/heat transfer/subcriticality, azimuthal length scales, and wave time scales)
    - cf. plane layer models

Radial velocity in the equatorial plane at $E = 10^{-6}$, $Ra/Ra_c = 10$, $Pm/Pr = 0.5$ (Yadav et al. 2016)
Convection-driven, spherical dynamo simulations

- **Greatly studied for the past decades** (e.g. Glatzmaier & Roberts 1995; Kageyama & Sato 1995; also reviews by Christensen & Wicht 2007; Jones 2011)
  - successful for reproducing observed features of planetary magnetic fields
  - a tool for understanding the dynamics with self-generated magnetic fields

- **MHD dynamos driven by Boussinesq convection in rotating spherical shells:**
  - Governing equations (dimensionless)
    \[
    \begin{align*}
    \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} &= \frac{Pm}{E} [2\hat{e}_z \times \mathbf{u} - \nabla p + (\nabla \times \mathbf{B}) \times \mathbf{B}] + \frac{Pm^2 Ra}{Pr} T \hat{e}_r + Pm \nabla^2 \mathbf{u} \\
    \frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T &= \frac{Pm}{Pr} \nabla^2 T - 1 \\
    \frac{\partial B}{\partial t} &= \nabla \times (\mathbf{u} \times \mathbf{B}) + \nabla^2 \mathbf{B} \\
    \nabla \cdot \mathbf{u} &= 0, \quad \nabla \cdot \mathbf{B} = 0
    \end{align*}
    \]
  - Parameters: modified Rayleigh, Ekman, kinetic/magnetic Prandtl numbers
    \[
    Ra = \frac{g\alpha |\epsilon| D^5}{\nu \kappa \eta}, \quad E = \frac{\nu}{\Omega D^2}, \quad Pr = \frac{\nu}{\kappa}, \quad Pm = \frac{\nu}{\eta}
    \]
    \[
    \sim 16 \text{ Ra}_{\text{crit}} = 10^{-4} - 10^{-6} \quad \text{Ra}_{\text{crit}} = 1 \quad \frac{\nu}{\eta} = 1-5
    \]
  - Leeds spherical dynamo code: based on pseudo spectral method (e.g. Jones et al. 2011)
Slow MR waves in dynamo simulations

• Slow modes identified:
  – retrograde drifts commonly seen in dynamo simulations
  – their speeds accounted for by total phase speeds of wave and mean flow advection, \((\omega_{MR} + \omega_{adv})/m\), where
    \[
    \hat{\omega}_{MR} = -\frac{\hat{\omega}_M}{\hat{\omega}_R} = -\frac{m^3(r_o^2 - s^2)\bar{B}_\phi^2}{2\rho\mu_0\Omega s^4} \\
    \omega_{adv} = \bar{\zeta}m = \frac{\bar{U}_\phi}{s}m
    \]
  – 2D spectral analysis is crucial to distinguish each component

• Note: wave contribution depends on the radius s
  – wave \(\sim<\) advection at larger s

\[\text{at } E = 10^{-5}, Pm/Pr = 5, Ra/Ra_c = 8 & \Lambda \sim 22\]
(Hori, Jones & Teed, 2015)
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  - retrograde drifts commonly seen in dynamo simulations
  - their speeds accounted for by **total phase speeds** of wave and mean flow advection, \((\omega_{MR} + \omega_{adv})/m\), where
    \[
    \frac{\hat{\omega}_{MR}}{\hat{\omega}_R} = -\frac{m^3 (r_o^2 - s^2) \langle B^2 \rangle}{2 \rho \mu_0 \Omega s^4}
    \]
    \[
    \omega_{adv} = \bar{\zeta} m = \langle \bar{U}_\phi \rangle / \bar{s} m
    \]
  - 2D spectral analysis is crucial to distinguish each component

- Note: wave contribution depends on the radius \(s\)
  - wave \(\sim<\) advection at larger \(s\)
Exploring more cases

- MR waves were found in models when torsional waves were found
  - generated magnetic fields of non-reversing dipole
  - for strong-field solutions ($\Lambda \sim 2$; $Pm \geq 5$ or $E \leq 10^{-5}$), good Taylorization (< 0.2), good geostrophy ($U'_c > 0.4$)

- Note: excited azimuthal wave-numbers $m$ vary
  - chosen by the convective instability
    - dependent on $E$, $Ra$, $\Lambda$, etc

\[ E = 5 \times 10^{-6}, Pm = 2, Ra/Ra_c = 16 \& \Lambda \sim 6 \]

\[ E = 10^{-4}, Pm = 5, Ra/Ra_c = 8 \& \Lambda \sim 22 \]
Nonlinearity on waveforms?

The observed waves illustrate

- no wave packets
- isolated, sharp waveforms
  - steepening
  - shifted to positive

- reminiscent of cnoidal/solitary waves in weakly nonlinear, dispersive waves (e.g. Whitham 1974)
  - cf. (nonmag) solitary Rossby (e.g. Redekkop 1977, Yamagata 1982)

Evolution of amplitude $<u_s'>$ at $s=0.5r_o$
(Hori, Teed & Jones 2017)
The role of nonlinear Lorentz force

- Coriolis and Lorentz terms are dominant in the axial vorticity eq.
  - Reynolds term remains minor

- The Lorentz term $\Xi_L$ can be expanded, in terms of the mean and fluctuating parts, as
  $$\Xi_L = \frac{P_m}{E} \left[ \langle \mathbf{B} \cdot \nabla j'_z \rangle + \langle b' \cdot \nabla j'_z \rangle + \text{(other terms)} \right]$$
  - first term for the restoring force
  - second term for the leading nonlinear part

- The sum of the dominant restoring and nonlinear terms reproduces steepened shapes
Toroidal field strength within the Earth’s core

• The dispersion relation tells us about waves riding on mean flow advection

\[ \hat{\omega}_{M\beta} = \omega - \omega_{\text{adv}} = -\frac{m^3(r_o^2 - s^2)\langle B^2 \rangle}{2\rho \mu_0 \Omega s^4} \]

– a geomagnetic drift speed of 0.56 °/yr at 40° S (Finlay & Jackson 2003)

– suppose a mean flow of 0.24 °/yr
  (Pais et al. 2015)

– Given m=5, this implies a z-mean toroidal field \( B_\phi \sim 12 \text{ mT} \) at s ~ 0.8r_o
  • equivalent to, or stronger than, the poloidal field \( B_s \geq 3 \text{ mT} \) (Gillet et al. 2010)

– constrains the dynamo mechanism?
  • e.g. \( \alpha^2 \)-type or \( \alpha\omega \) -type
  • stronger poloidal fields in dynamo simulations

(Hori, Jones & Teed, 2015)
In thin, stably stratified layers

- A stable layer at the top of the Earth’s core
  - SW models applied by poloidal field (Braginsky 1984, 1999)

- Solar tachocline at the bottom of the convection zone
  - SW models applied by toroidal field (Gilman 2000; Zaqarashvili et al. 2007)
  - ~3 m/s westward drifts and eastward wavetrains? (McIntosh et al. 2017)

Coronal brightpoints in Jan 2012 & at around 15° N / 22° S (McIntosh et al. 2017)
e.g. equatorial waves (cartesian)

- \( \beta \)-plane shallow water models applied by an azimuthal field

\[
\begin{align*}
\frac{\partial u_x}{\partial t} - fu_y & = \frac{B_x}{4\pi \rho} \frac{\partial b_x}{\partial x} - g \frac{\partial h}{\partial x}, \\
\frac{\partial b_x}{\partial t} & = B_x \frac{\partial u_x}{\partial x}, \\
\frac{\partial u_y}{\partial t} + fu_x & = \frac{B_x}{4\pi \rho} \frac{\partial b_y}{\partial x} - g \frac{\partial h}{\partial y}, \\
\frac{\partial h}{\partial t} + H_0 \left( \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} \right) & = 0,
\end{align*}
\]

- when \( f \sim \beta y \),

\[
\frac{d^2 u_y}{dy^2} + \left[ \frac{\omega^2}{C_0^2} - k_x^2 \left( 1 + \frac{v_A^2}{C_0^2} \right) - \frac{k_x \beta}{\omega (1 - k_x v_A^2 / \omega)^2} - \frac{\beta^2}{C_0^2 (1 - k_x v_A^2 / \omega) y^2} \right] u_y = 0
\]

- cf. nonmagnetic case (e.g. Matsuno 1966):
  - a Schroedinger eq.
  - oscillatory for \( |y| < y_c \), i.e. equatorially trapped waves

- In the presence of magnetic field
  - nonzero \( V_A \) increases \( y_c \), i.e. releasing the trapped waves
  - large \( V_A \) gives rise to a Bessel eq.
In spherical shells

- Nonaxisymmetric MAC waves classified:
  - inertio-gravity
  - Rossby
  - Kelvin

- Rossby: for eq. symmetric $B\phi = B_0 \sin \theta$
  (Marquez-Artavia et al., 2017)
  - fast modes
    - goes westward
    - in the limit $V_M^2/V_c^2 \ll 1$, $\omega = -\frac{2\Omega_0 m}{n(n+1)}$
  - slow modes
    - goes eastward
    - in the limit $V_M^2/V_c^2 \ll 1$, $\omega = \frac{m v_a^2}{2\Omega_0 R_0^2} (n(n+1) - 2)$.
    - slowly westward for $n=m=1$
  - even polar trapped at large $V_M^2/V_c^2$
  - become unstable at large $V_M^2/V_c^2$

Eigenfunctions of fast / slow MR waves
for $m=1$, $\alpha (\sim V_M^2/V_c^2) = 0.1$, $\epsilon^{-1} (\sim V_A^2/V_c^2) = 0.01$
Summary

• Geo-/Jovian dynamo simulations are supporting the excitation of magnetic Rossby waves for incompressible/anelastic fluids
  – crests/troughs going retrogradely on timescales of $O(10^{1-2} \text{ yrs})$ in the Earth’s core, about mean zonal flows
  – excited when torsional Alfvén waves were excited
    • for strong-field dynamos ($Pm \geq 5$ or $E \leq 10^{-4}$; $\Lambda \gtrsim 2$)
    – the speeds accounted for by the linear theory, but their waveforms steepened, likely due to nonlinear Lorentz terms
    – their speeds potentially revealing the strength of the ‘hidden’ toroidal field
    – induced by topography but also by compressibility
Thank you
QG vs. non-QG modes

- In spheres
  - e.g. for Malkus field (1967)
    \[ B_\phi = B_0 \sin \phi \]
  - the solution, \( P = P_n^m(\mu) P_n^m(\mu) \)
  - equatorially trapped for small \( n \)
  - even \((n-m)\): eq. symmetric (QG) modes
    - goes retrograde & faster (\( \approx -\omega_M^2/\omega_\beta \))
  - odd \((n-m)\): eq. anti-symmetric modes
    - goes prograde & slower (\( \approx +\omega_M^2/\omega_\iota \))

- cf. MC waves in simple plane layers
  - slow modes has no preference in propagation direction (\( \approx \pm \omega_M^2/\omega_C \))
  - The geometrical effect splits the modes into a faster & slower ones

Eigenfunctions for Malkus field (after Malkus 1967)